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Growth rate for the expected value of a generalized random Fibonacci sequence

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Abstract
We study the behaviour of generalized random Fibonacci sequences defined by the relation $g_n = |\lambda g_{n-1} \pm g_{n-2}|$, where the $\pm$ sign is given by tossing an unbalanced coin, giving probability p to the + sign. We prove that the expected value of g_n grows exponentially fast for any $0 < p \leqslant 1$ when $\lambda \geqslant 2$, and for any $p > (2-\lambda)/4$ when λ is of the form $2\cos(\pi/k)$ for some fixed integer $k \geqslant 3$. In both cases, we give an algebraic expression for the growth rate.

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1. Introduction

Random Fibonacci sequences have been introduced by Viswanath in [1] as a stochastic variation of the classical Fibonacci sequence: These are random sequences $(g_n)_n$ defined by their first two terms g_1 and g_2 (which in the following are assumed to be positive) and the recurrence relation

$$g_{n+1} = g_n \pm g_{n-1}, \tag{1}$$

where for each n the $\pm$ sign is chosen by tossing a balanced coin. Observe that since both signs have equal probabilities, the study of the linear induction (1) is equivalent to the nonlinear case, where we define the sequence by

$$g_{n+1} = |g_n \pm g_{n-1}|. \tag{2}$$

Indeed, in both cases we have $|g_{n+1}| = \|g_n| + |g_{n-1}\|$ or $\|g_n| - |g_{n-1}\|$, each with probability 1/2.

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A generalization of this notion consists in choosing the $\pm$ sign by an unbalanced coin (say, + with probability p and $-$ with probability $q := 1 - p$). Observe that, in contrast to what happens in the balanced setting, the study of the linear induction (1) is not in general equivalent to the nonlinear case (2).

A further generalization consists in fixing two real numbers, λ and μ, and considering the recurrence relation $g_n = \lambda g_{n-1} \pm \mu g_{n-2}$ (linear case) or $g_n = |\lambda g_{n-1} \pm \mu g_{n-2}|$ (nonlinear case, on which we concentrate in this paper), where the $\pm$ sign is chosen by tossing a balanced (or unbalanced) coin for each n. By considering the modified sequence $\tilde{g}_n := g_n/\mu^{n/2}$, we can always reduce to the case $\mu = 1$.

Random Fibonacci sequences are related to many fields, such as continued fractions, products of random matrices, condensed matter physics (see [2, 3] and references therein). In particular, the induction $g_n = \lambda g_{n-1} \pm \mu g_{n-2}$ can be interpreted in terms of the one-dimensional discretized Schrödinger equation.

In [4], we investigated the question of the almost-sure asymptotic growth rate of random Fibonacci sequences, in both the linear and the nonlinear cases. For $\lambda = \mu = 1$, we obtained an expression of the Lyapunov exponent which is simpler than that given by Viswanath in [1] and which is not restricted to $p = 1/2$: it is given by the integral of the natural logarithm over a specific measure (depending on p) defined on Stern–Brocot intervals. The corresponding results for random Fibonacci sequences with multiplicative coefficients, which involve not only some techniques presented here but also complementary considerations, are the object of another publication (see [5]).

In the present paper, we are concerned with the evaluation of the exponential growth rate for the expected value of random Fibonacci sequences. Note that by Jensen's inequality, we have $\mathbb{E}[|g_n|]^{1/n} \geqslant \mathbb{E}[|g_n|^{1/n}]$ where $\mathbb{E}$ stands for the expectation. Hence, the exponential growth rate for the expected value is generally larger than the almost-sure growth rate. We will see that in some cases, the expected value grows exponentially fast whereas the sequence almost surely decreases exponentially fast (see theorem 1 and [5]).

Surprisingly, our method is more easily applied in the nonlinear case, to which we restrict ourselves hereafter. Nevertheless, we believe that it can be adapted to the linear case (see section 9.3.1).

In the following, we refer to the random sequences $g_n = |\lambda g_{n-1} \pm g_{n-2}|$ where the + sign is chosen with probability p and the $-$ sign with probability $q := 1 - p$ as (p, λ)-*random Fibonacci sequences*.

This problem was studied by the second author in the particular case $p = 1/2$ and $\lambda = 1$ [6]: the growth rate of the expected value was proved to be asymptotically equal to $\alpha - 1 \approx 1.205\,569\,43$, where α is the only real zero of $\alpha^3 = 2\alpha^2 + 1$. The proof involves the study of the binary tree **T** naturally defined by the set of all $(1/2, 1)$-random Fibonacci sequences (if a node a has b as a child, then the node b has two children, labelled by $a + b$ and by $|a - b|$ respectively). The study of **T** is made by considering the biggest subtree of **T** (denoted by **R**) which shows no redundance, that is, in which we never see two different edges with the same values a and b (in this order) as the parent and child. This restricted tree **R** has many combinatorial and number-theoretic aspects which are of interest. Let us mention that, after the publication of [6], we realized that there was a problem in the last step of the proof of its main result; this mistake is explained in remark 1.

In the present paper, we correct and extend the result of [6] to some (p, λ)-random Fibonacci sequences: for any $p \in [0, 1]$, any $\lambda \geqslant 2$, and any λ of the form $2\cos(\pi/k)$ (denoted by λ_k), where $k \geqslant 3$ is an integer.

For $\lambda = \lambda_k$, the combinatorial properties of the tree **R** extend in a surprisingly elegant way, leading to an extremely natural generalization of the results previously mentioned. In particular, the link made in [4, 6] between the tree **R** and continued fraction expansion remains true for $\lambda_k = 2\cos(\pi/k)$ and corresponds to the so-called Rosen continued fractions, a notion introduced by David Rosen in [7]. We will not resort to continued fractions in the present work, but this aspect is presented in [5].

An interesting fact to note is that the values λ_k and $\lambda > 2$ are the only ones for which the group of transformations of the hyperbolic half-plane $\mathbb{H}^2$ generated by the transformations $z \longmapsto -1/z$ and $z \longmapsto z + \lambda$, said to be a *Hecke group*, is discrete (cf [8]). We will not use this fact in the following, but it suggests that it is highly probable that some link is to be made between random Fibonacci sequences and hyperbolic geometry; in particular, possible future extensions of the combinatorial point of view given by the restricted trees defined below could have some interpretation in hyperbolic geometry for values of λ for which the corresponding Möbius group is not discrete.

2. Results

Our main results are the following theorems.

Theorem 1. *Let $p \in [0, 1], k \geqslant 3$ and m_n be the expected value of the nth term of a (p, λ_k)-random Fibonacci sequence.*

- *If $p > p_c := (2 - \lambda_k)/4$, then*
$$\frac{m_{n+1}}{m_n} \underset{n\to\infty}{\longrightarrow} \alpha_k(p)\left(1 + \frac{pq^{k-1}}{\alpha_k(p)^k}\right) > 1,$$
where $\alpha_k(p)$ is the only positive root of the polynomial
$$P_k(X) := X^{2k} - \lambda_k X^{2k-1} - (2p-1)X^{2k-2} - \lambda_k pq^{k-1}X^{k-1} - p^2q^{2k-2}.$$
- *If $p = p_c$, then $(m_n)_n$ grows at most linearly.*
- *If $p < p_c$, then $(m_n)_n$ is bounded.*

Note that by Jensen's inequality, the critical value for the growth rate of the expected value is smaller than the critical value when one considers the almost-sure growth rate. It is proved in [5] that the latter is equal to $1/k$, which is strictly larger than $p_c = (2 - \lambda_k)/4 = (1 - \cos(\pi/k))/2$.

For $\lambda \geqslant 2$, we obtain the same exponential growth for the expected value of the nth term of a (p, λ)-random Fibonacci sequence in the nonlinear as in the linear case (that is, when we drop the modulus in the recurrence relation; see section 9.3.1). More precisely, we have the following result.

Theorem 2. *Let $\lambda \geqslant 2, 0 < p \leqslant 1$ and m_n be the expected value of the nth term of a (p, λ)-random Fibonacci sequence. Then*
$$\frac{m_{n+1}}{m_n} \underset{n\to\infty}{\longrightarrow} \frac{\lambda + \sqrt{\lambda^2 + 4(2p-1)}}{2}.$$

In view of the study of (p, λ)-random Fibonacci sequences for other values of λ, we also investigate an aspect of the regularity of the behaviour of the growth rate of such a sequence in the neighbourhood of $\lambda = 2$.

Corollary 1. *Let $0 < p \leqslant 1$. If we assume that, for any λ in the neighbourhood of 2, the expected value of a (p, λ)-random Fibonacci sequence increases exponentially fast with the growth rate equal to $\mathcal{G}(\lambda)$, then $\mathcal{G}$ cannot be analytic at $\lambda = 2$.*

In the particular case $p = 1/2$, we can even prove the non-analyticity of the growth rate on any left neighbourhood of 2.

Corollary 2. *Let $p = 1/2$. Assume that, for any $\lambda \leqslant 2$, the expected value of a $(1/2, \lambda)$-random Fibonacci sequence increases exponentially fast, with the growth rate equal to $\mathcal{G}(\lambda)$. If $\mathcal{G}$ is differentiable at $\lambda = 2$, then $\mathcal{G}'(2) = 1$. If $\mathcal{G}$ is of class C^n at $\lambda = 2$, then $\mathcal{G}^{(i)}(2) = 0$ for any $i \in [2, n]$. As a corollary, $\mathcal{G}$ cannot be analytic at $\lambda = 2$.*

We first prove theorem 2 in section 3, since the case $\lambda \geqslant 2$ is much easier. Section 4 is devoted to general facts about the reduced tree for $\lambda = \lambda_k$. It introduces notations and useful tools for the proof of theorem 1. This theorem is proved in sections 5–7, by extending the ideas introduced in [6] for the case $k = 3$ and $p = 1/2$. Proofs of corollaries 1 and 2 about the non-analyticity in the neighbourhood of 2 can be found in section 8. Section 9 contains a discussion about open questions.

3. Case $\lambda \geqslant 2$

We introduce the tree $\mathbf{T}_\lambda(a, b)$, which shows all different possible (p, λ)-random Fibonacci sequences with first positive terms $g_1 = a$ and $g_2 = b$. The root of the tree $\mathbf{T}_\lambda(a, b)$ is labelled by a, its unique child is labelled by b, and each node of the tree labelled by β and with the parent labelled by α has exactly two children, the right one labelled by $\lambda\beta + \alpha$ and the left one by $|\lambda\beta - \alpha|$. When β is the label of a node in $\mathbf{T}_\lambda(a, b)$ with the parent labelled by α, it will be convenient to consider the vector (α, β) as the label of the corresponding edge. We also talk about the right and left children of an edge labelled (α, β), which are respectively the edges labelled by $(\beta, \lambda\beta + \alpha)$ and $(\beta, |\lambda\beta - \alpha|)$.

We also introduce the weight of an edge in the tree $\mathbf{T}_\lambda(a, b)$, corresponding to the probability that a (p, λ)-random Fibonacci sequence passes through this edge. More formally, the initial edge in $\mathbf{T}_\lambda(a, b)$ has weight 1. If an edge has weight w, then its right and left children have respective weights pw and qw. (Recall that $q = 1 - p$.)

We organize the edges of the tree $\mathbf{T}_\lambda(a, b)$ in rows: the initial edge is the only edge in row 2; any child of an edge in row n is in row $n + 1$. For any $n \geqslant 2$, we denote by ψ_n the set of edges in row n in $\mathbf{T}_\lambda(a, b)$.

The *average* of any subset X of edges of $\mathbf{T}_\lambda(a, b)$ is defined as the sum $M(X)$ of all terms of the form βw, where β is the second coordinate of an edge in X and w is the weight of this edge. Observe that the expected value m_n of the nth term of a (p, λ)-random Fibonacci is given by $M(\psi_n)$.

It will be of interest to consider the sequence (ℓ_s) of numbers read along the leftmost branch of $\mathbf{T}_\lambda(a, b)$: $\ell_1 := a$, $\ell_2 := b$ and $\ell_{s+1} := |\lambda\ell_s - \ell_{s-1}|$ for $s \geqslant 2$.

We now turn to the proof of theorem 2.

Lemma 1. *If $b \geqslant a$ and $\lambda \geqslant 2$, then for any edge in $\mathbf{T}_\lambda(a, b)$ labelled by (α, β), we have $\beta \geqslant \alpha$.*

Proof. This is immediately proved by induction. □

Corollary 3. *If $b \geqslant a$ and $\lambda \geqslant 2$, then $M(\psi_n) = \lambda M(\psi_{n-1}) + (2p - 1)M(\psi_{n-2})$.*

Proof. We deduce from lemma 1 that absolute values are never used in the computations of labels in $\mathbf{T}_\lambda(a, b)$. By regrouping the contributions of the two children of an edge in row $n - 1$, and summing over row $n - 1$, we get the result. □

The preceding corollary shows that for $b \geqslant a$ and $\lambda \geqslant 2$, there exist constants C and C'(depending on a and b) such that

$$M(\psi_{n+2}) = C\alpha^n + C'\alpha'^{\,n},$$

where $\alpha := (\lambda + \sqrt{\lambda^2 + 4(2p-1)})/2$ and $\alpha' := (\lambda - \sqrt{\lambda^2 + 4(2p-1)})/2$ are the roots of the polynomial $X^2 - \lambda X - (2p-1)$. Observe that, by lemma 1, if $b \geqslant a$ then all labels in the tree $\mathbf{T}_\lambda(a, b)$ are larger than b; hence $M(\psi_{n+2}) \geqslant b > 0$. Since $\alpha > 1$ and $|\alpha'| < 1$, we deduce that $C > 0$, and theorem 2 is proved when $b \geqslant a$.

If $b < a$, we first consider the case where the sequence (ℓ_s) along the leftmost branch is unbounded. Then there exists a first $S \geqslant 2$ such that $\ell_{S+1} \geqslant \ell_S$. We can then consider $\mathbf{T}_\lambda(a, b)$ as the disjoint union of the trees $\mathbf{T}_\lambda(\ell_s, \lambda\ell_s + \ell_{s-1}), 2 \leqslant s \leqslant S$, and $\mathbf{T}_\lambda(\ell_S, \ell_{S+1})$. $M(\psi_{n+2})$ can be written as a convex combination of similar expressions in these trees. Since for each one of these trees, the labels of the first edge are well ordered, theorem 2 is valid for them. A simple computation shows that the result extends to $\mathbf{T}_\lambda(a, b)$.

It remains to consider the case where (ℓ_s) is bounded. We then consider $\mathbf{T}_\lambda(a, b)$ as the disjoint union of the leftmost branch and infinitely many trees, namely the trees $\mathbf{T}_\lambda(\ell_s, \lambda\ell_s + \ell_{s-1}), 2 \leqslant s$, whose first-edge labels are well ordered. For each s, there exist constants C_s and C'_s (depending on ℓ_{s-1} and ℓ_s) such that

$$M(\psi_{n+2}) = q^n \ell_{n+2} + \sum_{s=0}^{n-1} q^s p\left(C_s\alpha^{n-s+1} + C'_s\alpha'^{\,n-s+1}\right).$$

Using the fact that C_s and C'_s are bounded, we obtain that $M(\psi_{n+2}) \sim K\alpha^n$ for some $K > 0$, which ends the proof of theorem 2.

4. Reduced tree in the case $\lambda = \lambda_k$ for some $k \geqslant 3$

From now on, we fix an integer $k \geqslant 3$, and set $\lambda := \lambda_k$. We keep the notations concerning the tree $\mathbf{T}_\lambda(a, b)$ introduced in the preceding section. (See figure 1.)

We introduce the two matrices

$$L := \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix} \qquad \text{and} \qquad R := \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}. \tag{3}$$

Observe that the right child of an edge labelled (α, β) is labelled by $(\alpha, \beta)R$ and the left child is labelled by $(u, |v|)$ where $(u, v) = (\alpha, \beta)L$.

Definition 1. *Any right child in a tree is said to be a 0th left child. For any integer $m > 0$, a child in a tree is said to be an mth left child iff it is the left child of an $(m-1)$th left child.*

Proposition 1. *Let (α, β) be the label of an edge in $\mathbf{T}_{\lambda_k}(a, b)$. The $(k-1)$th left child of the right child of this edge is also labelled by (α, β).*

Proof. An elementary calculation shows that $L = PDP^{-1}$, where

$$D := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\pi/k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\pi/k} \end{pmatrix}, \qquad P := \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\pi/k} \\ 1 & \mathrm{e}^{-\mathrm{i}\pi/k} \end{pmatrix}.$$

As a consequence, we get that for any integer j,

$$RL^j = \frac{1}{\sin(\pi/k)} \begin{pmatrix} \sin\frac{j\pi}{k} & \sin\frac{(j+1)\pi}{k} \\ \sin\frac{(j+1)\pi}{k} & \sin\frac{(j+2)\pi}{k} \end{pmatrix}. \tag{4}$$

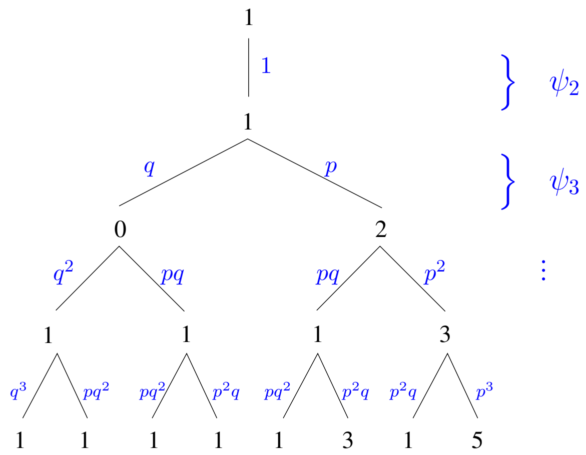

Figure 1. The beginning of the tree $\mathbf{T}_1(1, 1)$.

In particular, for $j \leqslant k-2$, RL^j has non-negative entries, and

$$RL^{k-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, for $j \leqslant k-2$, the jth left child of the right child of the edge labelled by (α, β) is labelled by $(\alpha, \beta)RL^j$ and the $(k-1)$th left child of the right child is labelled by $(\alpha, |-\beta|)$. □

We now define the tree $\mathbf{R}_k(a, b)$ as the subtree of $\mathbf{T}_{\lambda_k}(a, b)$ obtained by removing the left child of the initial edge and all the edges which are the $(k-1)$th left child.

Proposition 2 (Linearity in $\mathbf{R}_k(a, b)$). *Whenever it exists, the left child of an edge in $\mathbf{R}_k(a, b)$ labelled by (α, β) is labelled by $(\alpha, \beta)L$.*

Proof. By definition, $\mathbf{R}_k(a, b)$ contains only jth left children, for $0 \leqslant j \leqslant k-2$. The proposition is a direct consequence of the fact that for such j's, RL^j has non-negative entries. □

Considering $\mathbf{R}_k(a, b)$ as a subtree of $\mathbf{T}_{\lambda_k}(a, b)$, any edge in $\mathbf{R}_k(a, b)$ inherits its weight from $\mathbf{T}_{\lambda_k}(a, b)$. As for $\mathbf{T}_{\lambda}(a, b)$, we organize the edges of $\mathbf{R}_k(a, b)$ in rows, and denote by $\pi_n \subset \psi_n$ the set of edges in row n in $\mathbf{R}_k(a, b)$ (see figure 2).

In section 5, we will first consider $M(\pi_n)$, which is easier to study than $M(\psi_n)$. Then, the next step (section 6) will be to estimate $M(\psi_n)$ by partitioning the tree $\mathbf{T}_{\lambda_k}(a, b)$ in infinitely many copies of trees $\mathbf{R}_k(\ell_{s+1}, \ell_{s+2})$, where (ℓ_s) is the sequence of numbers read along the leftmost branch of $\mathbf{T}_{\lambda_k}(a, b)$: $\ell_1 := a$, $\ell_2 := b$ and $\ell_{s+1} := |\lambda_k \ell_s - \ell_{s-1}|$ for $s \geqslant 2$.

Remark 1. Let us explain why there is a mistake in the argument given in [6] (which deals with the particular case $k = 3$ and $p = 1/2$). The average growth rate is proved to be equal to some explicit value for two linearly independent pairs of initial values for the random Fibonacci sequence which are $(g_1, g_2) = (1, \varphi)$ and $(1, \varphi^{-1})$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. It is then asserted that, since the vectors $(1, \varphi)$ and $(1, \varphi^{-1})$ are linearly independent, any pair (a, b) can be written as a linear combination of $(1, \varphi)$ and $(1, \varphi^{-1})$ to get the tree $\mathbf{T}_{\lambda_3}(a, b)$ as a linear combination of the trees $\mathbf{T}_{\lambda_3}(1, \varphi)$ and $\mathbf{T}_{\lambda_3}(1, \varphi^{-1})$. The conclusion follows that, since both of these latter trees show the same growth rate, the growth

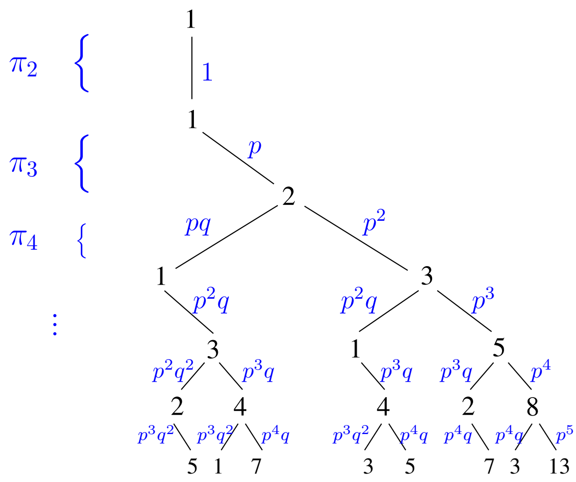

Figure 2. The beginning of the tree $\mathbf{R}_3(1,1)$, which is a subtree of $\mathbf{T}_{\lambda_3}(1,1)$.

rate of a random Fibonacci sequence does not depend on the initial values a and b (with the obvious restriction that $ab \neq 0$). In fact, this way of proving the theorem is wrong, as it can be easily seen by writing the equality $(1,1) = \varphi^{-2}(1,\varphi) + \varphi^{-1}(1,\varphi^{-1})$. The tree deduced from this linear combination is a tree with root 1 and child of the root equal to 1, but it does not correspond to $\mathbf{T}_{\lambda_3}(1,1)$ (since, for example, the left grandchild of the root is equal to $2\varphi^{-3}$ instead of 0).

5. Average $M(\pi_n)$ of row n of $\mathbf{R}_k(a,b)$

For any $n \geqslant 3$ and $0 \leqslant i \leqslant k-2$, we define π_n^i as the subset of π_n made of all its elements which are ith left children (π_n^0 thus corresponds to right children). We adopt the convention $\pi_2^{k-2} := \pi_2$ and $\pi_2^i := \emptyset$ for $i \leqslant k-3$. From the definition of $\mathbf{R}_k(a,b)$, we get that each node of the tree $\mathbf{R}_k(a,b)$ which has only one child is a $(k-2)$th left child.

5.1. A recursive formula for $M(\pi_n)$

The next lemma gives, for any $0 \leqslant i \leqslant k-2$, $M(\pi_n^i)$ as a function of $\left(M(\pi_{n-1}^j)\right)_{0\leqslant j\leqslant k-2}$ and $\left(M(\pi_{n-2}^j)\right)_{0\leqslant j\leqslant k-2}$.

Lemma 2. *For any integer $n \geqslant 4$, we have*

$$M(\pi_n^0) = \lambda p M(\pi_{n-1}) + pM(\pi_{n-2}) - pqM(\pi_{n-2}^{k-2}),$$
$$M(\pi_n^1) = \lambda q M(\pi_{n-1}^0) - pqM(\pi_{n-2}),$$
$$M(\pi_n^i) = \lambda q M(\pi_{n-1}^{i-1}) - q^2 M(\pi_{n-2}^{i-2}), \qquad 2 \leqslant i \leqslant k-2.$$

Proof. Consider an edge e in π_n whose parent has weight w and is labelled by (α,β).

Case 1. Assume that $e \in \pi_n^0$ (e is a right child). The contribution of e to $M(\pi_n^0)$ is $(\lambda\beta + \alpha)wp = \lambda p \cdot \beta w + p \cdot \alpha w$. Observe that when e runs over π_n^0, its parent runs over π_{n-1} and brings a contribution βw to $M(\pi_{n-1})$. Moreover, when e's parent runs over π_{n-1}^0, e's grandparent runs over π_{n-2} and has contribution $\alpha w/p$ to $M(\pi_{n-2})$. When e's parent runs over $\cup_{i=1}^{k-2}\pi_{n-1}^i$, e's grandparent runs over $\cup_{i=0}^{k-3}\pi_{n-2}^i$ and has contribution

$\alpha w/q$ to $M\left(\cup_{i=0}^{k-3}\pi_{n-2}^{i}\right) = M(\pi_{n-2}) - M\left(\pi_{n-2}^{k-2}\right)$. This proves the first equality of lemma 2.

Case 2. Assume that $e \in \pi_n^1$ (e is a left child). Its contribution to $M\left(\pi_n^1\right)$ is $(\lambda\beta - \alpha)wq = \lambda q \cdot \beta w - pq \cdot \alpha w/p$. Observe that e's parent is in π_{n-1}^0 (thus is a right edge), and brings a contribution βw to $M\left(\pi_{n-1}^0\right)$. Moreover e's grandparent is in π_{n-2}, has weight w/p and its contribution to $M(\pi_{n-2})$ is $\alpha w/p$. When e runs over π_n^1, its parent runs over π_{n-1}^0 and its grandparent runs over π_{n-2}.

Case 3. Assume that $e \in \pi_n^i$ for $2 \leqslant i \leqslant k-2$ (e is a left child). The contribution of e to $M\left(\pi_n^i\right)$ is $(\lambda\beta - \alpha)wq = \lambda q \cdot \beta w - q^2 \cdot \alpha w/q$. Observe that e's parent is in π_{n-1}^{i-1} (thus is a left edge), and brings a contribution βw to $M\left(\pi_{n-1}^{i-1}\right)$. Moreover e's grandparent is in π_{n-2}^{i-2}, has weight w/q and e is its only left grandchild in π_n^i. The contribution of e's grandparent to $M\left(\pi_{n-2}^{i-2}\right)$ is $\alpha w/q$. When e runs over π_n^i, its parent runs over π_{n-1}^{i-1} and its grandparent runs over π_{n-2}^{i-2}. This ends the proof of the lemma. □

In the proof of the preceding lemma, we only used the structure of the tree $\mathbf{R}_k(a, b)$ and the linear relation linking the labels of the edges in this tree, but not the specific value $\lambda = \lambda_k$. In the next lemmas, this specific value plays a central role.

Lemma 3. *We have for any $n \geqslant k+2$,*

$$qM\left(\pi_{n-2}^{k-3}\right) - \lambda M\left(\pi_{n-1}^{k-2}\right) = pq^{k-2}M(\pi_{n-k}).$$

Proof. Consider an edge e in π_{n-1}^{k-2} with weight w and label (α, β). Its parent is in π_{n-2}^{k-3} and has weight w/q. Thus, it contributes for $\alpha w/q$ to $M\left(\pi_{n-2}^{k-3}\right)$. Moreover, e's $(k-1)$th parent is in π_{n-k}, has weight w/pq^{k-2} and its label (u, v) is such that $(u, v)RL^{k-2} = (\alpha, \beta)$. Thus, $(u, v) = (\beta, \alpha - \lambda\beta)$ and this edge contributes for $(\alpha - \lambda\beta)w/pq^{k-2}$ to $M(\pi_{n-k})$. The conclusion follows from the fact that when e runs over π_{n-1}^{k-2}, its parent runs over π_{n-2}^{k-3} and its $(k-1)$th parent runs over π_{n-k}. □

Lemma 4. *We have for any $n \geqslant k+2$*

$$M\left(\pi_n^{k-2}\right) = pq^{k-2}\left(M(\pi_{n-k}) - qM\left(\pi_{n-k}^{k-2}\right)\right)$$

Proof. Consider an edge e in π_{n-k} with weight w and label (α, β). If $e \in \pi_{n-k}^j$, $j \leqslant k-3$, this edge is the ancestor of two edges in π_n^{k-2}, which have labels $(\alpha, \beta)RRL^{k-2}$ and $(\alpha, \beta)LRL^{k-2}$, and weights wp^2q^{k-2} and $wqpq^{k-2}$ respectively. Their contribution to $M\left(\pi_n^{k-2}\right)$ is thus βwpq^{k-2}. If $e \in \pi_{n-k}^{k-2}$, it is the ancestor of only one edge in π_n^{k-2}, having labels $(\alpha, \beta)RRL^{k-2}$ and weight wp^2q^{k-2}. Its contribution to $M\left(\pi_n^{k-2}\right)$ is thus βwp^2q^{k-2}. The conclusion follows from the fact that any edge in π_n^{k-2} has a unique ancestor in π_{n-k}. □

Lemma 5. *We have for any $n \geqslant k+2$,*

$$M(\pi_n) = \lambda M(\pi_{n-1}) + (2p-1)M(\pi_{n-2}) + pq^{k-1}M(\pi_{n-k}) + (q-p)qM\left(\pi_{n-2}^{k-2}\right).$$

Proof. Using lemmas 2 and 3, we get

$$M(\pi_n) = \sum_{i=0}^{k-2} M\left(\pi_n^i\right)$$
$$= \left(\lambda pM(\pi_{n-1}) + pM(\pi_{n-2}) - pqM\left(\pi_{n-2}^{k-2}\right)\right)$$

$$+\left(\lambda q M\left(\pi_{n-1}^{0}\right)-pqM(\pi_{n-2})\right)+\sum_{i=1}^{k-3}\lambda q M\left(\pi_{n-1}^{i}\right)-\sum_{i=0}^{k-4}q^2M\left(\pi_{n-2}^{i}\right)$$
$$=\lambda M(\pi_{n-1})-\lambda q M\left(\pi_{n-1}^{k-2}\right)+(p-pq-q^2)M(\pi_{n-2})$$
$$+(q^2-pq)M\left(\pi_{n-2}^{k-2}\right)+q^2M\left(\pi_{n-2}^{k-3}\right)$$
$$=\lambda M(\pi_{n-1})+(2p-1)M(\pi_{n-2})+(q-p)qM\left(\pi_{n-2}^{k-2}\right)+pq^{k-1}M(\pi_{n-k}).$$ □

In the particular case $p=q=1/2$, lemma 5 gives $M(\pi_n)$ as a function of $M(\pi_{n-1})$ and $M(\pi_{n-k})$. The next proposition gives a simple way to express $M(\pi_n)$ in terms of $(M(\pi_m))_{m<n}$ in the general case.

Proposition 3. *We have for any* $n\geqslant 2k+2$,

$$M(\pi_n)=\lambda M(\pi_{n-1})+(2p-1)M(\pi_{n-2})+pq^{k-1}\lambda M(\pi_{n-k-1})+p^2q^{2k-2}M(\pi_{n-2k}).$$

Proof. Applying twice lemma 5 (once for $M(\pi_n)$, then for $M(\pi_{n-k})$), the desired result follows from lemma 4. □

5.2. Study of $M(\pi_n)$

Lemma 6. *Let* $n\geqslant 2$ *and let* $Q(X)=X^n-\sum_{j=0}^{n-1}c_jX^j$, *where* $c_j>0$ *for all* $0\leqslant j\leqslant n-1$. *Then* Q *has a unique positive real root* α. *Moreover,* α *is of multiplicity 1 and all the other roots of* Q *have modulus strictly less than* α.

Proof. Since $Q(0)=-c_0<0$, Q has at least one positive real root α. Assume that $\beta\,\mathrm{e}^{\mathrm{i}\theta}\neq\alpha$ is another root of Q such that $\beta\geqslant\alpha$. We have

$$\beta^n=\left|\sum_{j=0}^{n-1}c_j\beta^j\,\mathrm{e}^{\mathrm{i}j\theta}\right|\leqslant\sum_{j=0}^{n-1}c_j\beta^j.$$

On the other hand,

$$\beta^n=\left(\frac{\beta}{\alpha}\right)^n\alpha^n=\sum_{j=0}^{n-1}c_j\beta^j\left(\frac{\beta}{\alpha}\right)^{n-j},$$

which gives a contradiction if $\beta>\alpha$. Now, if $\beta=\alpha$, we prove that $\left|\sum_{j=0}^{n-1}c_j\beta^j\,\mathrm{e}^{\mathrm{i}\theta}\right|=\sum_{j=0}^{n-1}c_j\beta^j$. Since $c_j>0$ for all $0\leqslant j\leqslant n-1$, this implies $\theta=2\ell\pi$ and contradicts the fact that $\beta\,\mathrm{e}^{\mathrm{i}\theta}\neq\alpha$.

If $Q(\alpha)=Q'(\alpha)=0$, then $n\sum_{j=0}^{n-1}c_j\alpha^j=n\alpha^n=\sum_{j=0}^{n-1}jc_j\alpha^j$, which is impossible since $c_j>0$ for all $0\leqslant j\leqslant n-1$. Therefore, α is a simple root. □

Lemma 7. *Let* $k\geqslant 3$ *be fixed and* $\lambda=\lambda_k$. *The polynomial*

$$P_k(X):=X^{2k}-\lambda X^{2k-1}+(1-2p)X^{2k-2}-\lambda pq^{k-1}X^{k-1}-p^2q^{2k-2}$$

has a unique positive real root, denoted by α_k, *which is of multiplicity 1.*

Moreover, if $p>p_c$, *all the other roots have modulus strictly less than* α_k. *If* $p\leqslant p_c$, P_k *has two conjugate roots of modulus* q, *its positive root* α_k *is smaller than* q *and all the other roots have modulus strictly less than* α_k.

Proof. We claim that $P_k(X)$ can be rewritten as the product of $X^2 - q\lambda X + q^2$ and

$$X^{2k-2} - pa_{2k-3}X^{2k-3} - p\sum_{j=0}^{k-3} a_{k-1+j}q^{k-3-j}X^{k-1+j} - p^2\sum_{j=0}^{k-2} a_j q^{2k-4-j}X^j,$$

where all coefficients $(a_j)_{0\leqslant j\leqslant 2k-3}$ are positive. Indeed, identifying terms of degree j for $0 \leqslant j \leqslant 2k-3$ yields to the following relations, from which we can inductively compute the coefficients a_j:

$$\begin{aligned}
a_0 &= 1\\
a_1 &= \lambda\\
a_j &= \lambda a_{j-1} - a_{j-2} \qquad \forall\, 2 \leqslant j \leqslant k-2\\
a_{k-1} &= \lambda + p(\lambda a_{k-2} - a_{k-3})\\
a_k &= \lambda a_{k-1} - pa_{k-2}\\
a_j &= \lambda a_{j-1} - a_{j-2} \qquad \forall\, k+1 \leqslant j \leqslant 2k-4\\
qa_{2k-3} &= \lambda a_{2k-4} - a_{2k-5}.
\end{aligned}$$

Observe that for $0 \leqslant i \leqslant k-3$, $(a_i, a_{i+1}) = (1, \lambda)L^i = (0, 1)RL^i$. Hence, we deduce from (4) that $a_i > 0$ for all $0 \leqslant i \leqslant k-2$. Note that (4) also gives

$$RL^{k-2} = \begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix}. \tag{5}$$

This allows us to write $(a_{k-3}, a_{k-2})L = (0, 1)RL^{k-2} = (1, 0)$, from which we get $a_{k-2} = 1$ and $\lambda a_{k-2} - a_{k-3} = 0$. Hence $a_{k-1} = \lambda > 0$, and then $a_k = \lambda^2 - p > 0$. Now, we have, for $1 \leqslant i \leqslant k-3$, $(a_{k-2+i}, a_{k-1+i}) = (p, \lambda)L^i = (q\lambda, p)RL^i$. Thus, using (4) once again, $a_j > 0$ for all $j \leqslant 2k-4$. Next, using (5), we can write

$$(a_{2k-5}, a_{2k-4})L = (q\lambda, p)RL^{k-2} = (q\lambda^2 + p, q\lambda),$$

and we get $\lambda a_{2k-4} - a_{2k-5} = q\lambda$. Hence we obtain $a_{2k-3} = \lambda > 0$, which proves that all the coefficients a_j are positive.

From the same equation above, we also get $a_{2k-4} = q\lambda^2 + p$, and it is then a simple computation to check that, in the product, terms of respective degree $2k-2$, $2k-1$ and $2k$ also coincide with those of P_k.

Since $X^2 - q\lambda X + q^2$ has two conjugate (nonreal) roots of modulus q, we conclude by using lemma 6 that P_k has a unique real positive root α_k, which is of multiplicity 1. Moreover, since $P_k(q) = q^{2k-2}(2 - \lambda - 4p)$, we have $\alpha_k > q$ if and only if $p > (2-\lambda)/4$. □

Let us denote by (β_j) the roots of the polynomial P_k with $\alpha_k = \beta_0$ and let β_1 and $\beta_2 = \overline{\beta_1}$ be the two conjugate (nonreal) roots of $X^2 - q\lambda_k X + q^2$. Thanks to proposition 3, for any $n \geqslant 0$, we have

$$M(\pi_{n+2}) = Q_0\beta_0^n + \sum_{j\neq 0} Q_j(n)\beta_j^n, \tag{6}$$

where Q_j is a polynomial (depending on a and b) of a degree strictly less than the multiplicity of the root β_j.

Moreover, since $M(\pi_{n+2})$ is a real number, the coefficient Q_0 is real and coefficients corresponding to pairwise conjugate roots are conjugate.

Lemma 8. *As soon as $(a, b) \neq (0, 0)$, the coefficient Q_0 is positive. If $p \leqslant p_c$, the roots of $X^2 - q\lambda_k X + q^2$ are simple roots of P_k. If $p < p_c$, the associated coefficients Q_1 and Q_2 are null.*

Proof. First assume that $p \leqslant p_c$. By lemma 7, the roots of $P_k/(X^2 - q\lambda_k X + q^2)$ are of modulus smaller than q. Hence, $\beta_1 = q\,\mathrm{e}^{\mathrm{i}\theta}$ and $\beta_2 = q\,\mathrm{e}^{-\mathrm{i}\theta}$ (with $\theta \neq 0$ mod. 2π) are simple roots of P_k. We also know that the associated coefficients are conjugate and that $\beta_0 < q$ when $p < p_c$. Hence, we deduce from (6) that when $p < p_c$, $Q_1 \neq 0$ implies

$$M(\pi_{n+2}) \sim 2q^n\,\mathrm{Re}(Q_1\,\mathrm{e}^{\mathrm{i}n\theta}).$$

But this contradicts the fact that $M(\pi_{n+2})$ is positive for all n.

If $p \geqslant p_c$, β_0 is the root of the largest modulus of P_k. Moreover, since the coefficients Q_1 and Q_2 are null whenever $p < p_c$, we have $M(\pi_{n+2}) \sim C_0\beta_0^n$ whenever $p \neq p_c$. If $p = p_c$, we have $\beta_0 = q$; thus

$$M(\pi_{n+2}) \sim q^n(Q_0 + 2\,\mathrm{Re}(Q_1(n)\,\mathrm{e}^{\mathrm{i}n\theta})).$$

If $(a, b) \neq (0, 0)$, since $M(\pi_{n+2})$ is positive for all $n \geqslant 1$, we conclude that $Q_0 > 0$ whatever p is. □

6. Average $M(\psi_n)$ of row n of $\mathbf{T}_{\lambda_k}(a, b)$, for $\lambda = \lambda_k, k \geqslant 3$

6.1. Leftmost branch of $\mathbf{T}_{\lambda_k}(a, b)$

We denote by $(\ell_n)_n$ the sequence corresponding to the 'left branch' of the tree $\mathbf{T}_{\lambda_k}(a, b)$, that is, the sequence defined as

$$\ell_1 := a \qquad \ell_2 := b \qquad \ell_n := |\lambda\ell_{n-1} - \ell_{n-2}| \quad (n \geqslant 3).$$

We have the following result.

Proposition 4. *The sequence* $(\ell_n)_n$ *is bounded.*

When $\lambda_k = 1$ ($k = 3$), it is easily seen that the sequence is upper bounded by $\max(a, b)$. Strangely enough, the general case is quite more difficult to apprehend. Note that we cannot use proposition 2 and $L^k = Id$ to say that this sequence is periodic (of period k) and thus bounded, since ℓ_n's belong to the left branch of the full tree $\mathbf{T}_{\lambda_k}(a, b)$, which is not contained in $\mathbf{R}_k(a, b)$.

We give here a proof based on a geometrical interpretation, which can be applied for any $0 < \lambda < 2$. This proof also appears in [5]. The key argument relies on the following observation. Let θ be such that $\lambda = 2\cos\theta$. Fix two points P_0, P_1 on a circle centred at the origin O, such that the oriented angle (OP_0, OP_1) equals θ. Let P_2 be the image of P_1 by the rotation of angle θ and centre O. Then the respective abscissae x_0, x_1 and x_2 of P_0, P_1 and P_2 satisfy $x_2 = \lambda x_1 - x_0$. We can then geometrically interpret the sequence (ℓ_n) as the successive abscissae of points in the plane.

Lemma 9 (existence of the circle)**.** *Let* $\theta \in\,]0, \pi[$*. For any choice of* $(x, x') \in \mathbb{R}_+^2 \backslash \{(0, 0)\}$*, their exist a unique* $R > 0$ *and two points M and M', with respective abscissae x and x', lying on the circle with radius R centred at the origin, such that the oriented angle* (OM, OM') *equals* θ*.*

Proof. Assume that $x > 0$. We have to show the existence of a unique R and a unique $t \in\,]-\pi/2, \pi/2[$ (which represents the argument of M) such that

$$R\cos t = x \qquad \text{and} \qquad R\cos(t + \theta) = x'.$$

This is equivalent to

$$R\cos t = x \qquad \text{and} \qquad \cos\theta - \tan t\sin\theta = \frac{x'}{x},$$

which obviously has a unique solution since $\sin\theta \neq 0$.

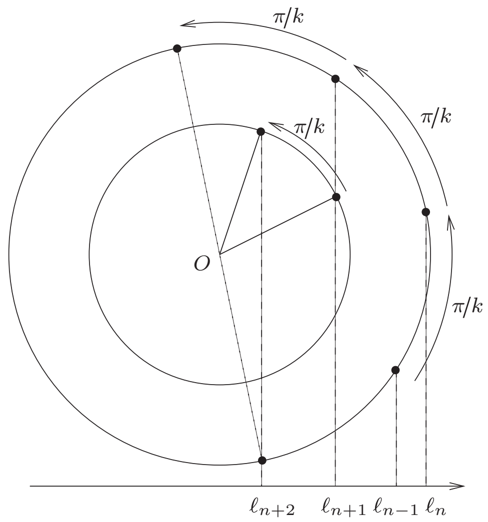

Figure 3. $R_n = R_{n+1}$ is the radius of the largest circle and R_{n+2} is the radius of the smallest.

If $x = 0$, the unique solution clearly is $R = x'/\cos(\theta - \pi/2)$ and $t = -\pi/2$.
Remark. Since $x_1 > 0$, we have $t + \theta < \pi/2$. □

Proof of proposition 4. At step n, we interpret ℓ_{n+1} in the following way. Applying the lemma with $x = \ell_{n-1}$ and $x' = \ell_n$, we find a circle of radius $R_n > 0$ centred at the origin and two points M and M' on this circle with abscissae x and x'. Consider the image of M' by the rotation of angle θ and centre O. If its abscissa is non-negative, it is equal to ℓ_{n+1}, and we will have $R_{n+1} = R_n$. Otherwise, we also have to apply the symmetry with respect to the origin to get a point with abscissa ℓ_{n+1}. The circle at step $n+1$ may then have a different radius, but we now show that the radius always decreases (see figure 3).

Indeed, denoting by α the argument of M', we have in the latter case $\pi/2 - \theta < \alpha \leqslant \pi/2$, $\ell_n = R_n \cos\alpha$ and $\ell_{n+1} = R_n \cos(\alpha + \theta + \pi) > 0$. At step $n+1$, we apply the lemma with $x = R_n \cos\alpha$ and $x' = R_n \cos(\alpha + \theta + \pi)$. From the proof of the lemma, if $\ell_n = 0$ (i.e. if $\alpha = \pi/2$), $R_{n+1} = R_n \cos(\alpha + \theta + \pi)/\cos(\theta - \pi/2) = R_n$. If $\ell_n > 0$, we have $R_{n+1} = R_n \cos\alpha/\cos t$, where t is given by

$$\cos\theta - \tan t \sin\theta = \frac{\cos(\alpha + \theta + \pi)}{\cos\alpha} = -(\cos\theta - \tan\alpha \sin\theta).$$

We deduce from the preceding formula that $\tan t + \tan\alpha = 2\cos\theta/\sin\theta > 0$, which implies $t > -\alpha$. On the other hand, as noted at the end of the proof of the preceding lemma, $t + \theta < \pi/2$; hence $t < \alpha$. Therefore, $\cos\alpha < \cos t$ and $R_{n+1} < R_n$.

Since $\ell_n \leqslant R_n \leqslant R_1$ for all n, the proposition is proved. □

Remark 2. The behaviour of the sequence (ℓ_n) strongly depends on the initial values a and b. It is proved in [5] that if a/b admits a finite λ-continued fraction expansion, the sequence (ℓ_n) is ultimately periodic. On the other hand, when ℓ_n decreases exponentially fast to 0, the exponent depends on the ratio a/b. Different examples of such behaviour are given in [5].

6.2. A formula for $M(\psi_n)$

For any $s \geqslant 0$ and $n \geqslant 2$, we denote row n of the tree $\mathbf{R}_k(\ell_{s+1}, \ell_{s+2})$ by $\pi_{n,s}$. In particular, $\pi_{n,0} = \pi_n$ is row n of the tree $\mathbf{R}_k(a, b)$.

Proposition 5. *For any $n \geqslant 0$,*

$$M(\psi_{n+2}) = \sum_{m=0}^{\lfloor n/k \rfloor} \sum_{s=0}^{n-km} c_{n,m} (pq^{k-1})^m q^s M(\pi_{n+2-s-km,s}),$$

where $c_{n,m} := \binom{n}{m} - (k-1)\binom{n}{m-1}$.

Proof. Let us code any trajectory from row 2 to row $n+2$ in the tree $\mathbf{T}_{\lambda_k}(a, b)$ by its successive steps $(X_i)_{3\leqslant i\leqslant n+2}$ ($X_i = R$ for a right step or L for a left step). Because of proposition 1, the label of the last edge of the trajectory does not change when we remove a pattern RL^{k-1} and its weight is divided by pq^{k-1}. By successively removing all patterns RL^{k-1} in $(X_i)_{3\leqslant i\leqslant n}$, we obtain a *reduced* sequence, that is, a subsequence which never contains $(k-1)$ successive L's, except eventually at the beginning. Moreover, its last edge has the same label as the last edge of the initial trajectory. Let us denote by s the number of left steps at the beginning of the reduced sequence and by m the number of removals. By removing these s left steps, we obtain a trajectory in the tree $\mathbf{R}_k(\ell_{s+1}, \ell_{s+2})$, whose last edge has the same label as the last edge initial trajectory and whose weight has been divided by $(pq^{k-1})^m q^s$.

Conversely, with a reduced sequence of length $n - km$ (for $0 \leqslant m \leqslant \lfloor n/k \rfloor$), we can associate a set of trajectories in $\mathbf{T}_{\lambda_k}(a, b)$ ending in row $n+2$ by successively adding m patterns RL^{k-1}. Denote by $c_{n,m}$ the cardinal of this set. To conclude the proof of the proposition, it remains to prove that $c_{n,m} := \binom{n}{m} - (k-1)\binom{n}{m-1}$. It is obviously true for all n when $m = 0$. We claim that the sequence $(c_{n,m})$ satisfies $c_{n+1,m} = c_{n,m} + c_{n,m-1}$ for any $n \geqslant 0$ and $1 \leqslant m \leqslant \lfloor (n+1)/k \rfloor$. Indeed, there are two disjoint classes of trajectories in $\mathbf{T}_{\lambda_k}(a, b)$ of length $n+1$ which can be obtained from a reduced sequence by successively adding m patterns RL^{k-1}.

(i) Those for which no pattern RL^{k-1} is inserted at the end. Their number is equal to $c_{n,m}$. Observe that this class is empty for $n = km$, since in this case, we start with an empty sequence.

(ii) Those for which at least one pattern RL^{k-1} is inserted at the end. They can be obtained by first inserting a pattern at the end and then by inserting $(m-1)$ other patterns anywhere but at the end of the new sequence. Their number is equal to $c_{n,m-1}$.

It is straightforward to check that the numbers $\binom{n}{m} - (k-1)\binom{n}{m-1}$ satisfy the same induction; thus, they coincide with $(c_{n,m})$. □

Remark 3. The coefficients $c_{n,m}$ can be obtained by generalizing Pascal's triangle (which corresponds to $k = 1$): we start by writing an infinite column of 1. On line k, we write a second 1 at the right of the first one. This new 1 is the beginning of a second column obtained by Pascal's rule: $c_{n+1,m} = c_{n,m} + c_{n,m-1}$. On line $2k$, that is, k lines after the beginning of the second column, we start a third column with the value $c_{2k,2} = c_{2k-1,1}$ and go on using Pascal's rule. Each new column starts k rows after the previous one; its first term is given by the rule $c_{pk,p} = c_{pk-1,p-1}$ (that is, the first term of a column is equal to the term of one row and one column before it) and the next terms of the column are given by Pascal's rule.

Here is the beginning of the triangle in the case $k = 4$:

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	1				
$n=1$	1				
	1				
	1				
$n=k$	1	1			
	1	2			
	1	3			
	1	4			
$n=2k$	1	5	4		
	1	6	9		
	1	7	15		
	1	8	22		
$n=3k$	1	9	30	22	
	1	10	39	52	
	1	11	49	91	
	1	12	60	130	
$n=4k$	1	13	72	190	130
	1	14	85	162	320
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

7. Proof of theorem 1

Recall that we denoted by (β_j) the roots of the polynomial P_k (studied in lemma 7), with $\alpha_k = \beta_0$ being the eigenvalue with the largest modulus. We deduce from (6) and proposition 5 that there exist polynomials $Q_{j,s}$ (depending on ℓ_{s+1} and ℓ_{s+2}) of a degree less than the multiplicity of β_j, such that for any $n \geqslant 0$,

$$M(\psi_{n+2}) = \sum_j \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m}(pq^{k-1})^m \sum_{s=0}^{n-km} q^s Q_{j,s}(n-km-s)\beta_j^{n-km-s}.$$

We now study the contribution of each β_j. We will find an equivalent of the contribution of the dominant root β_0 and prove that the contribution of other roots is negligible.

Lemma 10. *For any j, the coefficients of the polynomials $(Q_{j,s})_s$ are uniformly bounded with respect to s. Moreover, the coefficients $(Q_{0,s})_s$ are positive.*

Proof. This comes from proposition 4 and lemma 8. □

If $p > p_c$ (respectively if $p \leqslant p_c$), let $\epsilon > 0$ be small enough such that for all $j \geqslant 1$ (respectively $j \geqslant 3$), $\rho_j := |\beta_j|(1+\epsilon) < \beta_0$. The previous lemma implies that there exists a constant K such that for all n, $\left|Q_{j,s}(n)\beta_j^n\right| \leqslant K\rho_j^n$. The contribution of β_0 to $M(\psi_{n+2})$ can be written as $U_{n,0}(\beta_0)$, where

$$U_{n,0}(x) := x^n \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} \left(\frac{pq^{k-1}}{x^k}\right)^m \sum_{s=0}^{n-km} Q_{0,s}\left(\frac{q}{x}\right)^s.$$

The contribution of any other β_j can be bounded by $KU_n(\rho_j)$, where

$$U_n(x) := x^n \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} \left(\frac{pq^{k-1}}{x^k} \right)^m \sum_{s=0}^{n-km} \left(\frac{q}{x} \right)^s.$$

Observe that $U_{n,0}(x)$ is bounded by a constant times $U_n(x)$ since the coefficients $(Q_{0,s})_s$ are positive and bounded.

Proposition 6. *Let $x > 0$. We set $f(x) := x\big(1 + \frac{pq^{k-1}}{x^k}\big)$. There exists $C(x) > 0$, depending only on x, such that*

- *if $x > q$ and $x^k > (k-1)pq^{k-1}$, then $U_{n,0}(x) \sim C(x)(f(x))^n$ as $n \to \infty$;*
- *if $x > q$, then $|U_n(x)| \leqslant C(x)(f(x))^n$;*
- *$U_n(q) = O(n)$ as $n \to \infty$;*
- *if $x < q$, then $U_n(x) = O(1)$ as $n \to \infty$.*

Proof.
Case 1: $x > q$. When $j = 0$ (which corresponds to the dominant eigenvalue of P_k), the coefficients $(Q_{0,s})_s$ are positive and bounded, so we can choose S large enough such that

$$\left| \sum_{s=S+1}^{n-km} Q_{0,s} \left(\frac{q}{x} \right)^s \right| \leqslant \delta.$$

Set $A(x) := \sum_{s=0}^{S} Q_{0,s} \big(\frac{q}{x}\big)^s$. Hence,

$$A(x)x^n \sum_{m=0}^{\lfloor (n-S)/k \rfloor} c_{n,m} \left(\frac{pq^{k-1}}{x^k} \right)^m \leqslant U_{n,0}(x) \leqslant (A(x)+\delta)x^n \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} \left(\frac{pq^{k-1}}{x^k} \right)^m.$$

Observe that, since $c_{n,m} = \binom{n}{m} - (k-1)\binom{n}{m-1} = \binom{n}{m}\frac{n-km+1}{n-m+1}$, we can rewrite $\sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} \big(\frac{pq^{k-1}}{x^k}\big)^m$ as

$$\left(1 + \frac{pq^{k-1}}{x^k} \right)^n \mathbb{E} \left[\frac{1 - kM/n + 1/n}{1 - M/n + 1/n} \mathbb{1}_{\{M \leqslant \lfloor n/k \rfloor\}} \right],$$

where M is a binomial random variable with parameters (n, θ). Since $M/n \to \theta$ almost surely as $n \to \infty$, we get that this expectation goes to $\frac{1-k\theta}{1-\theta}$ if $\theta < 1/k$ (that is, if $x^k > (k-1)pq^{k-1}$), and the same is true if we replace n by $n - S$. Therefore, if $x^k > (k-1)pq^{k-1}$, for n large enough,

$$A(x) \left(\frac{1-k\theta}{1-\theta} - \delta \right) \leqslant \frac{U_{n,0}(x)}{x^n \big(1 + \frac{pq^{k-1}}{x^k}\big)^n} \leqslant (A(x)+\delta) \left(\frac{1-k\theta}{1-\theta} + \delta \right).$$

For $j \neq 0$, since $x > q$, $|U_n(x)|$ is bounded above, up to a multiplicative constant $C(x)$, by

$$x^n \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} \left(\frac{pq^{k-1}}{x^k} \right)^m \leqslant x^n \left(1 + \frac{pq^{k-1}}{x^k} \right)^n,$$

because $c_{n,m} \leqslant \binom{n}{m}$.

Case 2: $x = q$. We easily see that

$$U_n(q) = \sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} p^n q^{n-m} (n - km + 1) \leqslant n.$$

Case 3: $x < q$.

$$U_n(x) = x^n \sum_{m=0}^{\lfloor n/k \rfloor} \sum_{s=0}^{n-km} c_{n,m} \left(\frac{pq^{k-1}}{x^k} \right)^m \frac{(q/x)^{n-km+1} - 1}{q/x - 1},$$

which is, up to a multiplicative constant $C(x)$, less than

$$\sum_{m=0}^{\lfloor n/k \rfloor} c_{n,m} p^m q^{n-m} \leqslant 1.$$

□

Proof of theorem 1. Assume that $p > p_c$. We know from lemma 7 that $\beta_0 = \alpha_k > q$, and all the other roots β_j are such that $|\beta_j| < \alpha_k$.

We use proposition 6, and need to understand the variations of f. Elementary computations show that $f(x)$ decreases when x ranges from 0 to $x_{\min} := \sqrt[k]{(k-1)pq^{k-1}}$ and then increases. Observe that $f(q) = 1$; hence $f(x_{\min}) \leqslant 1$.

We claim that $\alpha_k > x_{\min}$. This is true if $p \leqslant 1/k$ because, in this case, $q \geqslant x_{\min}$. This remains true if $p > 1/k$. Otherwise, we would have $q < \alpha_k \leqslant x_{\min}$; hence $f(\alpha_k) < 1$. Moreover, if $|\beta_j| > q$, we also have $f(|\beta_j|) < 1$, and the contributions of other β_j's are, by proposition 6, at most linearly increasing with n. This would imply that $M(\psi_n) = O(n)$. But we know from [5] that when $p > 1/k$, the nth term of a (p, λ_k)-random Fibonacci sequence almost surely grows exponentially fast. By Jensen's inequality, this is all the more true for its expected value $M(\psi_n)$, so we get a contradiction.

It follows from proposition 6 that the contribution to $M(\psi_{n+2})$ of β_0 is $U_{n,0}(\beta_0) \sim C(\beta_0)(f(\beta_0))^n$ as $n \to \infty$, and that for $j \neq 0$, the contribution of β_j, which is bounded by $KU_n(\rho_j)$, is negligible with respect to $U_{n,0}(\beta_0)$. This ends the proof of theorem 1 in the case $p > p_c$.

Assume that $p = p_c$. We know from lemma 7 that $\beta_0 = q$, and all the other roots β_j are such that $|\beta_j| \leqslant q$. We thus deduce from proposition 6 that $M(\psi_n)$ grows at most linearly.

Assume that $p < p_c$. By lemma 8, we know that $(Q_{1,s})_s$ and $(Q_{2,s})_s$ are null. Moreover, we know from lemma 7 that $\beta_0 < q$ and that $\beta_j < \beta_0$ for all $j \geqslant 3$. Using proposition 6, we conclude that $M(\psi_n)$ is bounded. □

8. Non-analyticity in the neighbourhood of 2

Proof of corollary 1. By theorem 2, we know that for $\lambda \geqslant 2$, $\mathcal{G}(\lambda)$ is a root of the polynomial $Q_\lambda(X) := X^2 - \lambda X - (2p-1)$. Assuming that $\mathcal{G}$ is analytic in the neighbourhood of 2, we have $\mathcal{G}(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2 + 4(2p-1)}}{2}$ on this neighbourhood; thus, $\mathcal{G}(\lambda)$ is a root of Q_λ on this neighbourhood. On the other hand, theorem 1 says that for $\lambda = \lambda_k$ and $p > p_c$, $\mathcal{G}(\lambda_k) = \alpha_k(p)\left[1 + \frac{pq^{k-1}}{\alpha_k(p)^k}\right]$, where $\alpha_k(p)$ is a positive root of P_k. Thus, for any $k \geqslant 3$, we easily get that

$$\alpha_k(p)^{2k-2} Q_{\lambda_k}(\mathcal{G}(\lambda_k)) = 2pq^{k-1}(\alpha^k(p) + pq^{k-1}) > 0,$$

which proves that, for any $k \geqslant 3$, $\mathcal{G}(\lambda_k)$ is not a root of Q_{λ_k}. □

Proof of corollary 2. The growth rate of the expected value of a $(1/2, \lambda)$-random Fibonacci sequence has a very simple expression: it is equal to λ when $\lambda \geqslant 2$ (theorem 2) and to $2\alpha_k - \lambda_k$, where α_k is the only positive root of the polynomial $X^k - \lambda_k X^{k-1} - 1/2^k$, when $\lambda = \lambda_k$. Indeed, in the case $p = 1/2$, the polynomial $P_k(X)$ in theorem 1 can be rewritten as $(X^k + 1/2^k)Q_k(X)$, where $Q_k(X) := X^k - \lambda_k X^{k-1} - 2^{-k}$. Moreover, $Q_k(\alpha_k) = 0$ implies that $\alpha_k\left(1 + \frac{1}{2^k \alpha_k^k}\right) = 2\alpha_k - \lambda_k$.

Observe that $Q_k(\lambda_k) < 0$, which obviously implies that $\alpha_k > \lambda_k$. Since $Q_k(\alpha_k) = 0$, we have $\alpha_k^{k-1}(\alpha_k - \lambda_k) = 1/2^k$. Thus, $\alpha_k > \lambda_k \geqslant 1$ proves that $0 < \alpha_k - \lambda_k < 2^{-k}$.

Since λ_k tends to $\lambda_\infty = 2$ when k goes to infinity, if $\mathcal{G}'(2)$ exists, then we must have

$$\mathcal{G}'(2) = \lim_{k\to+\infty} \left(\frac{\mathcal{G}(\lambda_k) - \mathcal{G}(2)}{\lambda_k - 2} \right) = 1 + \lim_{k\to+\infty} 2\frac{\alpha_k - \lambda_k}{\lambda_k - 2}.$$

The numerator of the latter expression tends exponentially fast to 0, whereas the denominator is equivalent to $2\pi^2/k^2$, so we get $\mathcal{G}'(2) = 1$.

If $\mathcal{G}$ is of class C^2 at $\lambda = 2$, let us write its Taylor expansion at order 2:

$$\mathcal{G}(\lambda_k) = \mathcal{G}(2) + (\lambda_k - 2)\mathcal{G}'(2) + \frac{(\lambda_k - 2)^2}{2!}\mathcal{G}''(2) + O((\lambda_k - 2)^3).$$

We then get that $\mathcal{G}''(2)$ is equal to the limit of the ratio $2!2(\alpha_k - \lambda_k)/(\lambda_k - 2)^2$, which is equal to 0. The nullity of the nth derivative of $\mathcal{G}$ is obtained in a similar way, by an induction argument.

Hence, provided that $\mathcal{G}$ is of class C^∞ at $\lambda = 2, \mathcal{G}(2) = 2, \mathcal{G}'(2) = 1$ and $\mathcal{G}^{(n)}(2) = 0$ for any $n \geqslant 2$. The only possibility for $\mathcal{G}$ to be analytic at $\lambda = 2$ is to satisfy $\mathcal{G}(\lambda) = \lambda$ on a neighbourhood of 2. But we also have $\mathcal{G}(\lambda_k) = 2\alpha_k - \lambda_k$, which would imply $\alpha_k = \lambda_k$ for k large enough. This would contradict $P_k(\lambda_k) = -2^{-k} < 0$. □

9. Open questions

9.1. Critical value

Theorem 1 states that for $p = p_c$, the growth of $\mathbb{E}(g_n)$ is at most linear. The proof of this result uses the fact that the labels ℓ_n on the leftmost branch of $\mathbf{T}_{\lambda_k}(a, b)$ are bounded. It is proved in [5] that if a/b admits a finite λ-continued fraction expansion, the sequence (ℓ_n) is ultimately periodic. The arguments developed in the proof of proposition 6 show that, in this case, $\mathbb{E}(g_n)$ does grow linearly. However, we believe that for most choices of the initial values a and b, these labels decrease exponentially fast to zero, and that this ensures that $\mathbb{E}(g_n)$ is bounded.

9.2. Numerical simulation

As suggested to us by Steven Finch, from INRIA, numerical evidence of the growth rate of the expected value of a random Fibonacci sequence is not easy to obtain. This is due to the different behaviour of g_n and of $\mathbb{E}(g_n)$. For $\lambda = \lambda_k$, comparison with the result obtained in [5] shows that for $\frac{2-\lambda_k}{4} < p \leqslant 1/k$, the expected value of the nth term of a random Fibonacci sequence increases exponentially fast, whereas the sequence contains almost surely a bounded subsequence. When $1/k < p < 1$, numerical estimation of the growth rate of the expected value of g_n given by theorem 1 suggests that it is strictly greater than the almost-sure growth rate. This would imply that the variance of g_n increases exponentially fast with the growth rate at least twice the growth rate of the expected value since, by Jensen's inequality,

$$\begin{aligned}\frac{1}{n}\log \mathbb{E}[(g_n - \mathbb{E}(g_n))^2] &\geqslant \frac{1}{n}\mathbb{E}[\log(g_n - \mathbb{E}(g_n))^2] \\ &= \frac{1}{n}\mathbb{E}[\log(1 - g_n/\mathbb{E}(g_n))^2] + 2\frac{1}{n}\log \mathbb{E}(g_n).\end{aligned}$$

If the growth rate of the expected value of g_n is strictly greater than the almost-sure growth rate, the first term on the right-hand side goes to zero.

In [9] where the case $p = 1/2$ and $\lambda = 1$ is considered, the growth rate of the variance is proved to be equal to $1 + \sqrt{5}$. It would be of interest to know better about the exact value of the variance for any p and, more generally, about the moments of a higher order.

9.3. Generalization

9.3.1. Linear case. It is of course of interest to consider the *linear case*, given by the relation $g_{n+1} = \lambda g_n \pm g_{n-1}$. We then have the easy induction $\mathbb{E}(g_n) = \lambda\mathbb{E}(g_{n-1}) + (2p-1)\mathbb{E}(g_{n-2})$ for any λ. The corresponding polynomial has two real roots $(\lambda \pm \sqrt{\lambda^2 + 4(2p-1)})/2$, and we easily get an explicit expression of $\mathbb{E}(g_n)$ depending on the initial values. The question of interest in this setting is rather to study the exponential growth of $\mathbb{E}(|g_n|)$. The almost-sure growth rate of $|g_n|$ in the linear case is studied in [5] for $\lambda = \lambda_k$ ($k \geqslant 3$) and $\lambda \geqslant 2$, and turns out to be more difficult than in the nonlinear case. The analysis of $\mathbb{E}(|g_n|)$ is also more intricate. Although we can embed the tree $\mathbf{R}_k(a, b)$ in the tree of all possible sequences, this embedding is more complex than what we describe in the present paper. The main reason is that after the removal of a pattern RL^{k-1}, left and right children are exchanged. However, we think that our method can be adapted to the linear case.

9.3.2. Underlying structure. In the present paper, the underlying probabilistic structure is a Bernoulli scheme of parameter p. We believe that our method can be extended to some more general processes. In particular, it is of interest to investigate what happens if the signs are given by a quasi-periodic system (the coding of an irrational rotation on the circle). This leads to some interesting constructions which involve substitutions, and could be connected with numerous works on quasi-periodic structures (see e.g. [10, 11]).

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